ORIT HAZZAN

REDUCING ABSTRACTION LEVEL WHEN LEARNING ABSTRACT ALGEBRA CONCEPTS

ABSTRACT. How do undergraduate students cope with abstract algebra concepts? How should we go about researching this question? Based on interviews with undergraduate students and on written questionnaires, a theoretical framework evolved which could coherently account for most of the data. According to this theoretical framework, students' responses can be interpreted as a result of reducing the level of abstraction. In this paper, the theme of reducing abstraction is examined, based on three interpretations for *levels of abstraction* discussed in mathematics education research literature. From these three perspectives on abstraction, ways in which students reduce abstraction level are analyzed and exemplified.

1. INTRODUCTION

The research described in this paper strives to account for mental processes of undergraduate students as they solve problems in abstract algebra. As it turns out, mathematics education research literature becomes increasingly scarce when one moves from elementary school level, to high school level, and on to college level (cf. Selden and Selden, 1993; Thompson, 1993; Dreyfus, 1990, 1995). Even at college level, most of the research deals with pre-calculus, calculus, linear algebra and discrete mathematics (see Hazzan, 1995, for a comprehensive literature survey). Only recently has serious research been directed towards learning and teaching of abstract algebra. Papers comprising literature on abstract algebra learning can be roughly divided into two groups:

a) Teaching methods of abstract algebra (Pedersen, 1972; Lesh, 1976; Macdonald, 1976; Buchthal, 1977; Quadling, 1978; Lichtenberg, 1981; Simmonds, 1982; Freedman, 1983; Petricig, 1988; Thras and Walls, 1991; Leron and Dubinsky, 1995).
b) Learning, understanding and concept development in abstract algebra (Selden and Selden, 1987; Hart, 1994; Dubinsky, Dautermann, Leron and Zazkis, 1994; Hazzan, 1994; Leron, Hazzan and Zazkis, 1994, 1995; Hazzan and Leron, 1996; Brown, DeVries, Dubinsky and Thomas, 1997, Asiala, Dubinsky, Mathews, Morics and Oktaç, 1997; Asiala, Brown, Kleiman and Mathews, 1998).

Educational Studies in Mathematics **40:** 71–90, 1999.

The research described in this paper belongs to the second category. Research findings presented in Section 3 describe ways in which students deal with abstract algebra concepts by making these concepts mentally accessible. More specifically, the ways in which students conceive abstract algebra concepts are analyzed through the theme of reducing the level of abstraction. As it turns out, in many cases, reducing the level of abstraction is an effective strategy. However, sometimes it can be used inappropriately and becomes misleading.

The importance of learning abstract algebra is widely acknowledged. Gallian (1990) says that 'abstract algebra is important in the education of a mathematically trained person. The terminology and methodology of algebra are used ever more widely in computer science, physics, chemistry, and data communications, and of course, algebra still has a central role in advanced mathematics itself.' (p. xi). Instructors are aware of the importance of learning abstract algebra. At the same time, many of them report difficulties on the part of the students in understanding ideas they (the instructors) try to communicate. Thus, educators try to find ways to help students understand abstract algebra concepts and look for ways, *relevant for the students*, to introduce these concepts. For example, in the introduction to his book *Abstract Algebra,* Herstein (1986) argues that since '[t]here is little purpose served in studying some abstract algebra object without seeing some nontrivial consequences of the study, [we present in the book] interesting, applicable, and significant results in each of the systems we have chosen to discuss.' (p. viii). In a similar spirit, Gallian (1990) describes his approach for presenting abstract algebra ideas in a book: 'What I have attempted to do here is to capture the traditional spirit of abstract algebra while giving it a concrete computational foundation and including applications. I believe that students will best appreciate the abstract theory when they have a firm grasp of just what is being abstracted.' (p. xi). Gallian (ibid.) and Herstein (ibid.) refer to the learning of abstract algebra in the traditional lecture hall. Nowadays, several initiatives use programming languages (such as ISETL or Maple) to teach abstract algebra. For example, Dubinsky and Leron (1994), who use ISETL, attempt 'to help create an environment in which students construct, for themselves, mathematical concepts appropriate to understanding and solving problems in this area.' (p. xvii).

Despite these attempts to improve the teaching of abstract algebra, usually it is the first undergraduate mathematical course in which students 'must go beyond learning 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes (problems).' (Dubinsky et al., 1994, p. 268). Indeed, it is in the abstract algebra course that students are asked, for the first time, to deal with concepts which are introduced abstractly. That is, concepts are defined and presented by their properties and by an examination of 'what facts can be determined just from [the properties] alone.' (Dubinsky and Leron, 1994, p. 42). The new mathematical style of presentation leads students to adopt mental strategies which enable them to mentally cope with the new approach as well as with the new kind of mathematical objects. A plausible theoretical framework to explain students' ways of thinking in abstract algebra situations is the focus of this paper.

2. METHOD

As mentioned above, the research area of understanding abstract algebra concepts by undergraduate students is relatively uncharted. 'Now the growing body of research literature concerning the learning of collegiate mathematics contains a few studies focusing on abstract algebra.' (Asiala et al., 1998, p. 13). This trend is reflected mainly by Category (b) mentioned above. A specific framework for cognitive research and curriculum development of collegiate mathematics has guided several studies from this category. This framework is developed and being used by RUMEC-1, a subgroup of the larger Research in Undergraduate Mathematics Education Community (RUMEC). A complete discussion of this research framework is described in Asiala et al., 1996.

The research described in this paper examines students' understanding of abstract algebra from a different perspective. In this research, the *data* have guided the subsequent theoretical organization, in the spirit of 'grounded theory' (Glaser and Strauss, 1967). Glaser and Strauss explain that '[g]enerating a theory from data means that most hypotheses and concepts not only come from the data, but are systematically worked out in relation to the data during the course of the research.' (p. 6). This attitude, which intertwines development of a theory with the research process itself, is especially suitable in cases such as the one described in this paper in which a new field is investigated.

In order to gain a wide and varied picture on which to base the theoretical framework, the data were collected from a variety of sources. Some of these sources were planned (interviews, written research questionnaires, regular classroom tests, homework assignments), and others were incidental (observations in abstract algebra classes and occasional talks with students). Although not all of these sources are explicitly presented in the present report, they did help determine the directions and shape the ideas in the early stages of the study.

Semi-structured interviews were the main research tool. The interview questions appear in the Appendix. Nine undergraduate students attending standard abstract algebra lectures, based on the 'theorem-proof' format, were interviewed during their first course in abstract algebra.[1] The students were computer science majors in their freshmen year in a top-rank Israeli university. Prior to the abstract algebra course, the students had taken a linear algebra course, which had been taught in a rather abstract approach, stressing proofs and abstract vector spaces. Additive and multiplicative arithmetic groups, permutation groups, and groups of symmetries of a regular polygon were introduced as examples in lectures prior to the interviews. Each student took part in five weekly interviews, each interview lasting 1 hour. Questions and student data were translated from the original Hebrew for this paper.

The interviews focused on five fundamental abstract algebra subjects: groups, subgroups, cosets, Lagrange's theorem, and quotient groups. Questions presented in the interviews were relatively simple and mathematical knowledge required to solve them was relatively basic. Yet, in order to solve these questions students had to know the basic concepts, to examine relationships among them, and to analyze their properties. For example, question 24, 'Does a group of even order always have a subgroup of order 2?', was chosen for its potential to reveal students' thinking on groups, subgroups, and orders. At each stage during these five weeks, prior to the interview, the students had been introduced in the lecture to the concepts the interview focused on.

Here are some additional details about the other tools used for collecting data:

- Written research questionnaires: On several occasions, a written questionnaire was distributed among the members of the whole class in order to determine the prevalence of a certain phenomenon as it appeared in the interviews.
- Regular classroom tests and homework assignments: These tests and assignments reflected the expected requirements of the course and were not specifically designed for the research. However, upon reading the work of the students, it was observed that several phenomena repeatedly appeared.
- Incidental discussions: Some students came to office hours to ask questions. These questions often pointed to a student's conceptual difficulties and led to a brief impromptu interview on these difficulties.

3. ABSTRACTION AND WAYS IN WHICH STUDENTS REDUCE IT

This section describes the theoretical framework developed during the study, which aims to explain students' conceptions of abstract algebra concepts. The title of this framework is *reduction of the level of abstraction* (in short, *reducing abstraction*). From this perspective, most responses and conceptions on the part of students can be attributed to their tendency to work on a lower level of abstraction than the one in which concepts are introduced in class. The term 'reducing abstraction' should *not* be conceived as a mental process which necessarily results in misconceptions or mathematical errors. The mental process of reducing abstraction level indicates that students find ways to cope with new concepts they learn. They make these concepts mentally accessible, so that they would be able to think with them and handle them cognitively.

Abstraction is a complex concept which has many faces in general, and in especially in the context of mathematics and mathematics education. It has been dealt with by many psychologists and educators (e.g., Beth and Piaget, 1966). In the mathematics education research community it has been discussed from several perspectives (cf. Tall, 1991; Noss and Hoyles, 1996; Frorer, Hazzan and Manes, 1997). There is no consensus in regard to a unique meaning for abstraction. However, there is an agreement that the notion of abstraction can be examined from various perspectives, that certain types of concepts are more abstract than others, and that the ability 'to abstract' is an important skill for carrying out meaningful mathematics. Noss and Hoyles (1996) say: 'There is more than one kind of abstraction.' (p. 49). Consequently, as will be discussed, there is more than one way to reduce the level of abstraction.

The theme of reducing abstraction, presented in this paper, is based on three interpretations for *levels of abstraction* discussed in literature. That is, abstraction level as the quality of the relationships between the object of thought and the thinking person, abstraction level as reflection of the process-object duality, and abstraction level as the degree of complexity of the concept of thought. Relevant references for each interpretation appear later in the paper. It is important to note that these interpretations of abstraction are neither independent nor do they exhaust all possible interpretations for abstraction. As mentioned above, abstraction is a complex concept and it is not the purpose of the paper to review all its interpretations.[2] However, connections between the above three interpretations for abstraction are examined.

In the following, the three interpretations for abstraction are discussed. Analyses of the ways in which students think are examined in each case through the theme of reducing abstraction. In all cases, the reduction of abstraction level is a way of describing the phenomenon that in problem-solving situations students (and in fact all problem solvers) give the concepts involved some meaning. As it turns out, the meaning students give to some of the concepts can sometimes be interpreted as being on a lower level of abstraction than that in which concepts are introduced in class.

3.1. *Abstraction level as the quality of the relationships between the object of thought and the thinking person*

This interpretation for abstraction stems from Wilensky's (1991) assertion that whether something is abstract or concrete (or anything on the continuum between those two poles), is not an inherent property of the thing, 'but rather a *property of a person's relationship to an object.*' (p. 198). In other words, for each concept and for each person we may observe a different level of abstraction which reflects previous experiential connection between the two. The closer a person is to an object and the more connections he/she has formed to it, the more concrete (and the less abstract) he/she feels to it. Based on this perspective, some of students' mental processes can be attributed to their tendency to make an unfamiliar idea more familiar or, in other words, to make the abstract more concrete. This idea is described by Papert's metaphor of looking for familiar faces in an unfamiliar crowd:

> For me, getting to know a domain of knowledge [. . .] is much like coming into a new community of people. Sometimes one is initially overwhelmed by a bewildering array of undifferentiated faces. Only gradually do the individuals begin to stand out (Papert, 1980, p. 137).

The idea of making the unfamiliar familiar in abstract algebra context is illustrated in this paper by students' tendency to base arguments, which refer to general groups, on numbers and number operations – familiar mathematical objects, with which they have had previous mathematical experience since elementary school. Selden and Selden (1987), suggest that '[e]ven [the] limited collection of examples [presented at abstract algebra courses at the junior level] is rich enough for students to see that not everything behaves the way the more naive students expect. What they expect is really a misconception, namely, that the rules they know for dealing with real numbers are universal.' (p. 462).

Students' tendency to rely on systems of numbers, when asked to solve problems about other groups, can be explained by one of the basic ideas of constructivism. That is, that new knowledge is constructed based on existing knowledge. Thus, unknown (hence abstract) objects and structures are constructed based on existing mental structures. More specifically, if a given problem requires the manipulation of mental objects (like groups) the students haven't yet constructed, they would tend to think with familiar objects (like numbers), and to solve the problem using these familiar entities instead.

The following two examples illustrate how students reduce abstraction level by basing their arguments on mathematical entities they are familiar with – numbers and number operations.

Example 1:
Groups of numbers (such as Z in relation to addition or R-{0} in relation to number multiplication) are only some of the examples students come across in standard abstract algebra courses. Since each of these groups possesses many additional mathematical properties such as commutativity and ordering, none is a typical example of a general group. Generic examples of groups, which students come across in abstract algebra courses, are permutation groups and groups of symmetries of regular polygons. From the data analysis it turns out that students often treat groups as if they were made only of numbers and of operations defined on numbers.

In the following excerpt, Tamar tries to determine whether Z_3 (i.e., the set {0, 1, 2} with addition mod 3) is a group. She takes for granted that 'the inverse' of 2 is $^1/_2$ (as with rational numbers in relation to number multiplication), and tries to locate it in Z_3. Since she does not find $^1/_2$ in Z_3 she concludes that Z_3 is not a group:

[Z_3] is not a group. Again I will not have the inverse. I will not have one half. I mean, 1 over. . . I mean, if I define this [Z_3] with multiplication, I will not have the inverse for each element. [. . .] It will not be in the set. [. . .] I'm trying to follow the definition. [. . .] What I mean is that I know that I have the identity, 1. What I have to check is if I have the inverse of each. . . I mean, I have to see whether I have the inverse of 2 and I know that the inverse of 2 is $^1/_2$. [. . .] Now, one half, and on top of that do mod 3 [. . .], then it is not included. I mean, I don't have the inverse of . . . My inverse is not included in the set. Then it's not a group. I do not have the inverse for each element.

In this excerpt Tamar automatically considers the group operation as number multiplication and thus changes the operation from a less familiar one (addition mod 3) to a more familiar one. This may be a result of the use of the term 'multiplication' for a general binary operation in the group definition, together with the need, mentioned above, to mentally hang on to some familiar entity. However, it is worthwhile to notice that Tamar uses the abstract theoretic-terms (identity, inverse) correctly, and since a single axiom failure denies a group, her explanation (had to be taken into account in respect to the group operation she considered) makes sense.

Example 2:
In many cases, the use of surface clues may help greatly in problem solving. But, the conclusions thus reached must be carefully checked. If this is not done, the tendency to use surface clues may be quite misleading, as in the following misuse of Lagrange's theorem.

In a written questionnaire (but not in the interviews), 113 abstract algebra students were presented with the following question:

In an exam a student wrote: 'Z_3 is a subgroup of Z_6'.
In your opinion, is this statement true, partially true, or false? Please explain your answer.

Incorrect answers ('the statement is true') were given by 73 students, 20 of whom invoked Lagrange's theorem, in essentially the following manner:

According to Lagrange's theorem Z_3 is a subgroup of Z_6 because 3 divides 6.[3]

By giving this answer, students not only confuse Lagrange's theorem with its converse (*If* $k \mid o(G)$*, then there exists in G a subgroup of order k* – which is not a true statement), but actually use an *incorrect* version of the converse statement (*If* $o(H) \mid o(G)$ *then H is a subgroup of G*). Thus, a combination of several logic-based confusions, students have with mathematical deductions, appear to exist here. A detailed analysis of this phenomenon is given in Hazzan and Leron (1996).

However, in the context of this paper, I suggest that the popularity of Lagrange's theorem in this context is derived from its strong link to mathematical entities students are familiar with, i.e., numbers and number divisibility. At the stage the problem was presented to the students, many of them felt mentally disconnected from the notions appearing in the question (the groups Z_3 and Z_6, and the concept of subgroup). At the same time, the *numbers* 3 and 6, were familiar and meaningful to the students and, moreover, students could easily find a relationship between the two, which led to a sensible solution to the problem. In short, students were guided by the appearance of the numbers 3 and 6 and the fact that 3 divides 6, ignoring the other details in the problem. From the point of view of the interpretation of abstraction discussed in this section, it seems that students hang onto familiar mathematical entities, ignoring the meaning of the situation described in the problem. It helps them work and think with more familiar concepts or, in other words, make the abstract more concrete; that is, reducing the level of abstraction.

There seems to be a similarity with findings about elementary school pupils solving 'story problems' (Nesher and Teubal, 1975; Nesher, 1980; Schoenfeld, 1985). Elementary school pupils often look for two numbers and for a word that will clue them as to whether the operation involved is addition or subtraction. Then they go on giving the solution by performing the operation on these numbers, without even looking at the rest of the story.

3.2. *Abstraction level as reflection of the process-object duality*

In this section the idea of reducing abstraction is reviewed based on the process-object duality, suggested by some theories of concept development in mathematics education (Beth and Piaget, 1966; Dubinsky, 1991; Sfard, 1991, 1992; Thompson, 1985). Some of these theories, such as the APOS (action, process, object and scheme) theory, suggest a more elaborated hierarchy (cf. Dubinsky, 1991). However, for the discussion in this paper it is sufficient to focus on the process-object duality.

Theories that discuss this duality distinguish between *process conception* and *object conception* of mathematical notions. Dubinsky (1991) captures the passage from the first conception to the second one as a ‘conversion of a (dynamic) process into a (static) object’. *Process* conception implies that one regards a mathematical concept ‘as a potential rather than an actual entity, which comes into existence upon request in a sequence of actions.’ (Sfard, 1991, p. 4). When one conceives of a mathematical notion as an *object*, this notion is captured as one ‘solid’ entity. Thus, it is possible to examine it from various points of view, to analyze its relationships to other mathematical notions and to apply operations on it. According to these theories, when a mathematical concept is learned, its conception as a process precedes – and is less abstract than – its conception as an object (Sfard, 1991, p. 10). Thus, process conception of a mathematical concept can be interpreted as on a lower level of abstraction that its conception as an object; that is, abstraction level is reduced.

The mental process that leads from process conception to object conception is one mechanism Piaget named *reflective abstraction.* Dubinsky’s work (1991) tells us about the importance of this mechanism for mathematical thinking: ‘Piaget considered that it is reflective abstraction in its most advanced form that leads to the kind of mathematical thinking by which form or process is separated from content and that processes themselves are converted, in the mind of the mathematician, to objects of content.’ (p. 98).

This perspective of levels of abstraction is related to (and may even arise from) the previous interpretation for abstraction described in Section 3.1 – abstraction level as the quality of the relationships between the object of thought and the thinking person. The more one works with an *unfamiliar* concept initially being conceived as a *process*, the more *familiar* one becomes with it and may proceed toward its conception as an *object*.

Several studies analyze the understanding of abstract algebra concepts from the more elaborated perspective, that is, the APOS theory. For example, Brown et al. (1997) analyze the understanding of the notions binary operations, groups and subgroups; Asiala et al. (1997) present an analysis of students' understanding of cosets, normality and quotient groups; Asiala et al. (1998) focus on the development of students' understanding of permutations and symmetries.

My contribution to the discussion is with two additional aspects of process conception: (a) students' personalization of formal expressions and logical arguments by using first-person language, and (b) students' tendency to work with canonical procedures in problem solving situations. In the following examples, I explain how these aspects reflect process conception, and how they can be interpreted as mental processes of reducing abstraction.

Example 3:

This example illustrates how students personalize formal expressions and logical arguments by using first-person language. It further suggests that this way of speaking reflects process conception.

The use of first person language is especially pronounced in quantified expressions, where students often replace 'there exists' with 'I can find'. This may be explained by difficulties students experience with quantifiers (Dubinsky, Elterman and Gong, 1988; Harnik, 1986; Leron, Hazzan and Zazkis, 1994; Dubinsky, 1997), and may also be related to the fact that quantifiers rarely exist in high school mathematics (pun intended). Here, Dan, for example, explains how he checks the existence of inverses in question 3 (see Appendix):

Dan: I want to check if *I can find* for each... If for each element in the set *I can find* one element in the set [...] *I want to find* an inverse so that if *I multiply*, *I will* get *e*. (emphasis added).

The use of first-person language reflects a feeling of 'me doing something' and thus may be interpreted as process conception of the concepts the students think with. In the spirit of reducing abstraction, it is suggested that the students apply mental strategy that makes the unfamiliar mathematical language more familiar for them. Indeed, 3,500 years of the development of mathematics are intertwined with first-person language and recipes (cf. Kleiner, 1991). Thus, if students were asked in a one-semester course to conceive concepts that have been organized into a coherent theory over long periods of time, they (the students) have to find ways to cope with the unfamiliar terminology they face.

Here is one more example: Commutativity is a property of a set and of an operation defined on the set elements. When one presents a statement such as: 'Operation $\otimes$ is commutative for all subsets of a set *A*', we may say that commutativity is conceived as an object. In the following excerpt, in contrast, Adam defines (non)commutativity by describing *his own* actions:

Adam: Commutativity – that means, that if *I* ...[...] when *I replace* the place of the elements in the product then *I get* a different result. When *I change* the order in the multiplication *I get* a different result, so they will not be commutative. (emphasis added).

The use of first person language often goes together with very detailed and specific descriptions. In the following excerpt for example, Tanya examines the definition of quotient group $G/H = \{ Hx \mid x \text{ in } G\}$ in her notebook and explains it:

Tanya: I take all the elements [of *H*] and multiply them on the right with some element from *G*. I choose an element from *G* and I multiply all the elements of *H* on the right. [...] I mean, I should do it for all... *I do not do it together*. [...] *Each one by itself.* [...] So, I have to choose *a* and *x*. (emphasis added).

Tanya's description reflects process conception of a quotient group: she describes how *she builds* the cosets *one by one*, and how each coset is constructed by a sequence of products. Such a description enables her to know exactly where she is in the process of the construction, which coset she has already built and which ones are still ahead. By following this path students do not have to capture the whole situation at a glance as a single entity – hence they are not obliged to adopt an object conception. From the process-object duality point of view, this interpretation suggests that the level of abstraction is reduced.

Example 4:

This example illustrates students' tendency to work with canonical procedures in problem solving situations. This tendency can be explained by the fact that some of the concepts, and the relationships among them, are conceived as processes.

The term *canonical procedure* refers to a procedure that is more or less automatically triggered by a given problem. This can happen either because the procedure is naturally suggested by the nature of the problem, or because prior training has firmly linked a specific kind of problem with a specific procedure. The availability of a canonical procedure enables students to solve problems without analyzing properties of mathematical concepts, and to automatically follow the step-by-step algorithm the canonical procedure provides. In the following quote, for example, Guy presents a clear analysis of the structure of the cosets in a group, but then ‘forgets’ to use that knowledge; in the solution of another problem, he chooses the alternative of a messy coset calculation. Here is his description:

Guy: OK, it is known that equivalent classes divide the set into disjoint sets, the whole set. And now, if a coset is like an equivalent class then the cosets divide the . . . divide the group into disjoint sets. We know that in each coset the number of elements is equal.

In spite of this clear description, when asked to calculate the cosets of the subgroup { *1, 2, 4* } in Z_7-{ *0* } with multiplication mod 7, Guy first calculated all six cosets (one for each element of the group). It was only when he noticed that there were just two different cosets, that he began to look for the reason. Based on his clear description of the cosets as equivalent classes with the same number of elements, it is reasonable to assume that had he stopped for just a minute before jumping into a calculation mode, in order to try to examine the objects in the question, he would have known right away that there were only two different cosets.[4] However, Guy did not use his theoretical knowledge as a tool; instead, his first tendency was to start calculating the cosets by using a well-known procedure for this kind of task. This immediate response was done unconsciously: Guy did not consider the two alternatives (a use of the fact that cosets are equivalent classes vs. a use of a canonical procedure), choosing to use the canonical procedure. It is more plausible to assume that Guy *unconsciously* followed the path that the canonical procedure outlined him.

This example suggests that even in cases where students have a relatively advanced conception of mathematical notions, sometimes they tend to use a (less abstract) canonical procedure at the cost of many calculations. Indeed, it is much easier to make sense of a canonical procedure than of an abstract argument, which usually captures in one sentence several concepts together with relationships among them. Here, we can see that still, there is a gap between object conception of separated concepts and the ability to mentally link them all together into one schema conceived as an object.

3.3. *Abstraction level as the degree of complexity of the concept of thought*

This section examines abstraction by the degree of complexity of mathematical concepts. For example, a set of groups is a more compound object than one specific group in that set. Thus, the set of additive groups of prime order is a more compound mathematical entity than the group $[Z_5, +_5]$. It does not imply automatically, of course, that it should be more difficult to think in terms of compound objects. The working assumption here is that the more compound an entity is, the more abstract it is. In this respect, this section focuses on how students reduce abstraction level by replacing a set with one of its elements, thus, working with a less compound object. As it turns out, this is a handy tool when one is required to deal with compound objects that haven't yet been fully constructed in one's mind.

There is a connection between the interpretation for abstraction suggested in Section 3.2 (that is, abstraction as reflection of the process-object duality) and the one suggested here. This connection ties the *set* concept together with object conception and process conception: When the set concept is conceived as an *object*, a person becomes capable of thinking about it as a whole 'without feeling an urge to go into details.' (Sfard, 1991, p. 19); when conceived as a process, one conceives the set concept as a process in which its elements are grouped. Thus, when one deals with the elements of a set instead of with the set itself we may interpret this as *process* conception of the set concept.

The following example illustrates the tendency to deal with an element in a set instead of with the whole set. More specifically, it describes how students deal with a specific group when asked to deal with a set of groups containing it.

Example 5:
This example describes a student who faced a problem concerning a set of groups, and turned to consider only one group from the whole set (cf. also Rumelhart, 1989, p. 301: reasoning by example). In other words, he replaced a set with one of its elements. Considering a specific case may be a positive and helpful heuristic, as recommended by Polya (1973): 'Specialization is passing from the consideration of a given set of objects to that of a smaller set, or of just one object, contained in the given set. Specialization is often useful in the solution of problems.' (p. 190). The role of such specialization is to guide towards a general solution. However, there are cases in which students do use this heuristic, presenting an answer based on an analysis of a specific case, and do *not* return to the general case. Sometimes they do not go back to the general case simply because they are unable to do that – the mental structures needed to deal with the general case have not yet been constructed.

Question 24 in the interview asks: 'Does a group of even order always have a subgroup of order 2?'. In the following excerpt Ron describes how he 'just misses the point' and thus turns to work with a specific group. Apparently, he encounters difficulties in thinking about the family of groups of even order or difficulties in thinking about a generic group of even order.

Ron: A group of even order. . . does it always have a subgroup of order 2? [. . .] [pause] Why yes or why no? I just feel that I miss the point. [. . .] A group of even order can also be a group. . . let's say of order 6 for example. A group of even order? [. . .] Yes. If the order is 6 so it can have subgroups [of order] either 2 or 3, of order 2 or order 3.

4. Summary

This paper presents various mental processes by which students reduce abstraction level in abstract algebra problem-solving situations. As it turns out, in many cases, the mental mechanism of reducing abstraction helps students to cope successfully with problems presented to them. This theoretical framework corresponds with constructivist theories (cf. Kilpatrick, 1987; Sinclair, 1987; Davis, Maher and Nodding, 1990; Confrey, 1990; Smith, diSessa and Roschelle, 1993) and with the Piagetian terminology of assimilation and accommodation (Piaget, 1977). Given the abstraction level in which abstract algebra concepts are usually presented to students in lectures, and the lack of time for activities which may help students grasp these concepts, many of the students fail in constructing mental objects for the new ideas and in assimilating them with their existing knowledge. The mental mechanism of reducing the level of abstraction enables students to base their understanding on their current knowledge, and to proceed towards mental construction of mathematical concepts conceived on higher level of abstraction.

Acknowledgement

This work is based on the Ph.D. thesis of the author, written under the supervision of Prof. Uri Leron. I would like to express my gratitude to Uri Leron and Rina Zazkis for their useful remarks on an earlier draft of this paper.

Appendix: The interview

Remarks:

a) The following questions served only as a guide to the interviewer and were presented to the students orally.
b) The questions are grouped here in five parts according to their focus. However, the topics were not told to the students and the passage from one topic to the following one was done without any declaration about the current topic under the discussion.

Part 1: The concept of group (questions 1–6)

1. What is a group?

2. Can you give an example of a commutative group and of a non-commutative group?
3. Does the set {*a, b, c*} form a group relative to the following operation table?

$*$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

4. a) Is the identity element in a group unique?
   b) What does it mean that the identity element is unique?
   c) Prove that the identity element in a group is unique.
5. Does a group with only one element exist?
6. Let's construct an operation table of four elements *a, b, c, d*. Please fill it in such a way that it will form an operation table of a group of order 4.

Part 2: The concept of subgroup (questions 7–17)

7. What is a subgroup?
8. Can you give an example of a group *G* and a subgroup of *G*?
9. Consider the groups $(Z_3, +_3)$, $(Z_6, +_6)$. Is Z_3 a subgroup of Z_6?
10. Can you give an example of a subgroup of Z_6?
11. How do you check that nonempty subset of a group is a subgroup?
12. Please prove that the example of subgroup you gave in question 8, is indeed a subgroup.
13. Consider a group *G* and define the following subset of *G*:
    $C = \{a \text{ in } G \mid \forall x \text{ in } G, ax = xa\}$. Can you describe *C*'s elements?
14. The set *C* defined in question #13 is called the *center* of the group *G*. How would you check if a group element is (or is not) in the center of the group?
15. How will you check whether 3 is in the center of the group [*Z*, +]?
16. What is the center of a commutative group?
17. Prove that the center of a group *G* is a subgroup of *G*.

Part 3: The concept of coset (questions 18–21)

18. What is a coset?
19. What is the number of elements in every coset?
20. Consider the group $[Z_7\text{-}\{0\}, *_7]$ and its subgroup $[\{1, 2, 4\}, *_7]$. What are the cosets of the subgroup in the group?
21. What are the cosets of *2Z* in Z?

Part 4: Lagrange's theorem (questions 22–26)

22. Does $[Z_6, +_6]$ have a subgroup of order 4?
23. Does a group of order 12 always have a subgroup of order 6?
24. Does a group of even order always have a subgroup of order 2?
25. Can a group of order 7 have an element of order 4?
26. a) How many non trivial subgroups does a group of prime order have?
    b) Can you prove your last claim?

Part 5: The concept of quotient group (questions 27–30)

27. a) What is a quotient group?
    b) When does a quotient group exist?
    c) Can you state a property of a group that guarantees that all its subgroups are normal?
28. Is the following statement true or false?
    If G is a group and H is a commutative subgroup of G, then H is a normal subgroup of G.
29. Let G be $[Z_6, +_6]$ and H its subgroup $[\{ 0, 3 \}, +_6]$.
    a) Is H a normal subgroup of G?
    b) What are the elements of the quotient group G/H?
    c) What is the operation defined on the quotient group? (How do we multiply two elements in the quotient group?)
    d) Can you construct the operation table of G/H?
    e) Does this group remind you of another group?
30. What is the identity element in a quotient group?

NOTES

1. As mentioned before, in the last several years there has been a trend (especially in the USA) to teach abstract algebra with the programming language ISETL (Dubinsky and Leron, 1994; Leron and Dubinsky, 1995). Understanding of abstract algebra concepts by students who learned through this method are described in the following papers: Brown et al., 1997, Asiala et al., 1997; Asiala et al., 1998. I also experienced the teaching of abstract algebra through this method (together with Prof. Uri Leron) during 1993–1994. However, my intention in the study presented in this paper was to construct a theoretical framework to describe the understanding of abstract algebra concepts by students who attend standard lectures.
2. For example, there is the association of 'general' with 'abstract'. However, I agree with Staub and Stern (1997) who suggest that '[t]he degree of generality with respect to the range of (potential) real world references is one aspect of the abstract nature of quantitative schemata; but, [. . .] generality is not the core aspect that makes mathematical constructs especially abstract' (p. 65). Similarly, Leron (1987) says that '[i]n

mathematics, abstraction is closely related to generalization, but each can also occur without the other'.

3. Lagrange's theorem states that in a finite group the order of a subgroup divides the order of the group. (The *order* of a group is the number – finite or infinite – of its elements.) A sophisticated answer might claim that Z_3 *is* a subgroup of Z_6 because it is isomorphic to a subgroup of Z_6 (i.e., {0, 2, 4} with addition mod 6).
4. Actually, since there are only two distinct cosets, there is no need to do *any* calculations whatever: one coset must be the subgroup {1, 2, 4} itself, and the other one must be its complement {3, 5, 6}.

REFERENCES

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D. and Thomas, K.: 1996, 'A framework for research and curriculum development in undergraduate mathematics education', *Research in Collegiate Mathematics Education II*, The American Mathematics Society, pp. 1–32.

Asiala, M., Brown, A., Kleiman, J. and Mathews, D.: 1998, 'The development of students' understanding of permutations and symmetries', *International Journal of Computers for Mathematical Learning* 3(1), 13–43.

Asiala, M., Dubinsky, E., Mathews, D. M., Morics, S. and Oktaç, A.: 1997, 'Development of students' understanding of cosets, normality, and quotient groups', *Journal of Mathematical Behavior* 16(3), 241–309.

Beth, E. W. and Piaget, J.: 1966, *Mathematical Epistemology and Psychology*, D. Reidel Publishing Company, Dordrecht, The Netherlands.

Brown, A., DeVries, D. J., Dubinsky, E. and Thomas, K.: 1997, 'Learning binary operations, groups and subgroups', *Journal of Mathematical Behavior* 16(3), 187–239.

Buchthal, D. C.: 1977, 'Operative and nonoperative group definitions', *Mathematics Teacher* 70(3), 262–263.

Confrey, J.: 1990, 'Chapter 8: What constructivism implies for teaching', in R. B. Davis, C. A. Maher and N. Noddings (eds.), *Journal for Research in Mathematics Education, Monograph Number 4, Constructivist Views on the Teaching and Learning of Mathematics*, The National Council of Teachers of Mathematics, Inc., pp. 107–122.

Davis, R. B., Maher, C. A. and Noddings, N.: 1990, 'Chapter 12: Suggestions for the improvement of mathematics education', in R. B. Davis, C. A. Maher and N. Noddings (eds.), *Journal for Research in Mathematics Education, Monograph Number 4, Constructivist Views on the Teaching and Learning of Mathematics*, The National Council of Teachers of Mathematics, Inc., pp. 187–191.

Dreyfus, T.: 1990, 'Advanced mathematical thinking', in P. Nesher and J. Kilpatrick (eds.), *Mathematics and Cognition: A research Synthesis by the International Group for the Psychology of Mathematical Education*, Cambridge University Press, pp. 113–134.

Dreyfus, T.: 1995, 'Guest editorial, special issue – Advanced mathematical thinking', *Educational Studies in Mathematics* 29, 93–95.

Dubinsky, E., Elterman, F. and Gong, C.: 1988, 'The student's construction of quantification', *For the Learning of Mathematics* 8(2), 44–51.

Dubinsky, E.: 1991, 'Reflective abstraction in advanced mathematical thinking', in D. Tall (ed.), *Advanced Mathematical Thinking*, Kluwer Academic Press, pp. 95–123.

Dubinsky, E.: 1997, 'On learning quantification', *Journal of Computers in Mathematics and Science Teaching* 16(2/3), 335–362.

Dubinsky, E., Dautermann, J., Leron, U. and Zazkis, R.: 1994, 'On learning fundamental concepts of group theory', *Educational Studies in Mathematics* 27, 267–305.

Dubinsky, E. and Leron, U.: 1994, *Learning Abstract Algebra with Isetl*, Springer-Verlag.

Freedman, H.: 1983, 'A way of teaching abstract algebra', *American Mathematical Monthly* 90(9), 641–644.

Frorer, P., Hazzan, O. and Manes, M.: 1997, 'Revealing the faces of abstraction', *The International Journal of Computers for Mathematical Learning* 2(3), 217–228.

Gallian, J. A.: 1990, *Contemporary Abstract Algebra, second edition*, D. C. Heath and Company.

Glaser, B. and Strauss, A. L.: 1967, *The Discovery of Grounded Theory: Strategies for Qualitative Research*, Chicago, Aldine.

Harnik, V.: 1986, 'Infinitesimals from Leibniz to Robinson, Time to bring them to school', *The Mathematical Intelligencer* 8(2), 41–47, 63.

Hart, E.: 1994, 'A conceptual analysis of the proof-writing performance of expert and novice students in elementary group theory', in J. Kaput and E. Dubinsky (eds.), *Research Issues in Mathematics Learning: Preliminary Analyses and Results*, MAA notes, No. 33.

Hazzan, O.: 1994, 'A students' belief about the solutions of the equation $x=x^{-1}$ in a group', *Proceedings of the 18th international conference for the Psychology of Mathematics Education*, Lisbon, Portugal, Vol. III, pp. 49–56.

Hazzan, O.: 1995, *Undergraduate Students' Understanding of Abstract Algebra Concepts*. Ph.D. thesis, Technion – Israel Institute of Technology. Unpublished (Hebrew).

Hazzan, O. and Leron, U.: 1996, 'Students' use and misuse of Mathematical Theorems: The Case of Lagrange's Theorem', *For the Learning of Mathematics* 16, 23–26.

Herstein, I. N.: 1986, *Abstract Algebra*, Macmillan Publishing Company.

Kilpatrick, J.: 1987, 'What constructivism might be in mathematics education', *Proceedings of the Eleventh International Conference for the Psychology of Mathematical Education*, Volume I, 3–27.

Kleiner, I.: 1991, 'Rigor and proof in mathematics: A historical perspective', *Mathematics Magazine* 64(5), 291–314.

Leron, U.: 1987, 'Abstraction barriers in mathematics and computer science', *Proceedings of the Third International Conference for the Psychology of Mathematical Education*, Montreal.

Leron, U. and Dubinsky, E.: 1995, 'An abstract algebra story', *The American Mathematical Monthly* 102(3), 227–242.

Leron, U., Hazzan, O. and Zazkis, R.: 1994, 'Students constructions of group isomorphism', *Proceedings of the Eleventh International Conference for the Psychology of Mathematical Education* 3, 152–159.

Leron, U., Hazzan, O. and Zazkis, R.: 1995, 'Learning group isomorphism: A crossroads of many concepts', *Educational Studies in Mathematics* 29, 153–174.

Lesh, R. A.: 1976, 'The influence of two types of advanced organizers on an instructional unit about finite groups', *Journal for Research in Mathematics Education* 7(2), 87–91.

Lichtenberg, D. R.: 1981, 'A group whose elements are functions', *Mathematics Teacher* 74(7), 521–523.

Macdonald, I. D.: 1976, 'Modern algebra in the nineteenth century', *Australian Mathematics Teacher* 32(1), 33–38.

Nesher, P. and Teubal, E.: 1975, 'Verbal cues as an interfering factor in verbal problem solving', *Educational Studies in Mathematics* 6, 41–51.

Nesher, P.: 1980, 'The stereotyped nature of school word problems', *For the Learning of Mathematics* 1(1), 41–48.

Noss, R. and Hoyles, C.: 1996, *Windows on Mathematical Meanings – Learning Cultures and Computers*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Papert, S.: 1980, *Mindstorms – Children, Computers and Powerful Ideas*, Basic Books, Inc., Publishers / New York.

Pedersen, J. J.: 1972, 'Sneaking up on a group', *Two Year College Mathematics Journal* 3(1), 9–12.

Petricig, M.: 1988, 'Combining individualized instruction with the traditional lecture method in a college algebra course', *Mathematics Teacher* 81, 385–387.

Piaget, J.: 1977, 'Problems of Equilibration', in M. H. Appel and L. S. Goldberg (eds.), *Topics in Cognitive Development, Volume 1: Equilibration: Theory, Research and Application*, Plenum Press, NY, pp. 3–13.

Polya, G.: 1973, *How to Solve it?*, (Second edition), Princeton University Press, Princeton, New Jersey.

Quadling, D.: 1978, 'A contorted isomorphism', *Mathematics Teaching* 85, 48–49.

Rumelhart, D. E.: 1989, 'Toward a microstructural account of human reasoning', in S. Vosniadou and A. Ortony (eds.), *Similarity and Analogical Reasoning*, Cambridge University Press, pp. 298–312.

Schoenfeld, A. H.: 1985, *Mathematical Problem Solving*, Academic press, Inc.

Selden, A. and Selden, J.: 1987, 'Errors and misconceptions in college level theorem proving', *Proceedings of the Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics*, Cornell University, Vol. III, pp. 456–471.

Selden, A. and Selden, J.: 1993, 'Collegiate mathematics education research: What would that be like?' *The College Mathematics Journal* 24(5), 431–445.

Sfard, A.: 1991, 'On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin', *Educational Studies in mathematics* 22, 1–36.

Sfard, A.: 1992, 'Operational origins of mathematical objects and the quandary of reification – The case of function', in E. Dubinsky and G. Harel (eds.), *The Concept of Function – Aspects of Epistemology and Pedagogy*, MAA Notes.

Simmonds, G.: 1982, 'Computer discovery of a theorem of modern algebra', *Mathematics and Computer Education* 16(1), 58–61.

Sinclair, H.: 1987, 'Constructivism and the psychology of mathematics', *Proceedings of the Eleventh International Conference for the Psychology of Mathematical Education,* Volume I, pp. 28–41.

Smith, J. P., diSessa, A. A., Roschelle, J.: 1993, 'Misconceptions reconceived: A constructivist analysis of knowledge in transition', *The Journal of the Learning Sciences* 3(2), 115–163.

Staub, F. C. and Stern, E.: 1997, 'Abstract reasoning with mathematical constructs', *International Journal of Educational Research* 27(1), 63–75.

Tall, D.: 1991 (ed.), *Advanced Mathematical Thinking*, Kluwer Academic Press, Dordrecht, The Netherlands.

Thompson, P. W.: 1985, 'Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula', in E. A. Silver (ed.), *Teaching and*

Learning Mathematical Problem Solving: Multiple Research Perspective, Hillsdale, NJ, pp. 189–236.

Thompson, P. W.: 1993, ‘Yes, Virginia, some children do grow up to be mathematicians’, *Journal for Research in Mathematics Education* 24(3), 279–284.

Thrash, K. R. and Walls, G. L.: 1991, ‘A classroom note on understanding the concept of group isomorphism’, *Mathematics and Computer Education* 25(1), 53–55.

Wilensky, U.: 1991, ‘Abstract meditations on the concrete and concrete implications for mathematical education’, in I. Harel and S. Papert (eds.), *Constructionism*, Ablex Publishing Corporation, Norwood, NJ, pp. 193–203.

Department of Science Education – Technion,
Israel Institute of Technology,
Haifa 32000,
Israel
E-mail: oritha@techunix.technicon.ac.il