

ON THE CALCULUS OF RELATIONS

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The logical theory which is called the *calculus of (binary) relations*, and which will constitute the subject of this paper, has had a strange and rather capricious line of historical development.¹ Although some scattered remarks regarding the concept of relations are to be found already in the writings of medieval logicians, it is only within the last hundred years that this topic has become the subject of systematic investigation. The first beginnings of the contemporary theory of relations are to be found in the writings of A. De Morgan, who carried out extensive investigations in this domain in the fifties of the Nineteenth Century. De Morgan clearly realized the inadequacy of traditional logic for the expression and justification, not merely of the more intricate arguments of mathematics and the sciences, but even of simple arguments occurring in every-day life; witness his famous aphorism, that all the logic of Aristotle does not permit us, from the fact that a horse is an animal, to conclude that the head of a horse is the head of an animal. In his effort to break the bonds of traditional logic and to expand the limits of logical inquiry, he directed his attention to the general concept of relations and fully recognized its significance. Nevertheless, De Morgan cannot be regarded as the creator of the modern theory of relations, since he did not possess an adequate apparatus for treating the subject in which he was interested, and was apparently unable to create such an apparatus. His investigations on relations show a lack of clarity and rigor which perhaps accounts for the neglect into which they fell in the following years.

The title of creator of the theory of relations was reserved for C. S. Peirce. In several papers published between 1870 and 1882, he introduced and made precise all the fundamental concepts of the theory of relations and formulated and established its fundamental laws. Thus Peirce laid the foundation for the theory of relations as a deductive discipline; moreover he initiated the discussion of more profound problems in this domain. In particular, his investigations made it clear that a large part of the theory of relations can be presented as a calculus which is formally much like the calculus of classes developed by G. Boole and W. S. Jevons, but which greatly exceeds it in richness of expression and is therefore incomparably more interesting from the deductive point of view.

Peirce's work was continued and extended in a very thorough and systematic way by E. Schröder. The latter's *Algebra und Logik der Relative*, which appeared in 1895 as the third volume of his *Vorlesungen über die Algebra der Logik*, is so

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¹ For detailed historical information and thorough bibliographical references, see C. I. Lewis, *A survey of symbolic logic*, Berkeley, California, 1918.

far the only exhaustive account of the calculus of relations. At the same time, this book contains a wealth of unsolved problems, and seems to indicate the direction for future investigations.

It is therefore rather amazing that Peirce and Schröder did not have many followers. It is true that A. N. Whitehead and B. Russell, in *Principia mathematica*, included the theory of relations in the whole of logic, made this theory a central part of their logical system, and introduced many new and important concepts connected with the concept of relation. Most of these concepts do not belong, however, to the theory of relations proper but rather establish relations between this theory and other parts of logic: *Principia mathematica* contributed but slightly to the intrinsic development of the theory of relations as an independent deductive discipline. In general, it must be said that—though the significance of the theory of relations is universally recognized today—this theory, especially the calculus of relations, is now in practically the same stage of development as that in which it was forty-five years ago.

The fact just mentioned was one of the chief motives of my selecting the theory of relations as the subject for this talk. I shall confine myself here entirely to the theory of binary relations, since this is the only branch of the general theory which is at all developed. Moreover I shall be interested exclusively in that part of the theory of binary relations which is known as the calculus of relations; and I shall indeed be concerned merely with the calculus of finite operations on relations, as they were introduced by Peirce. I should like to acquaint you with two different methods of setting up the foundations of this elementary calculus in a rigorously deductive way, and I should like to discuss, or at least to formulate, some metalogical problems concerning this calculus.

The first method I am going to consider here consists in constructing the calculus of relations as a part of a more comprehensive logical theory, which corresponds approximately to the restricted functional calculus as it was given, for example, by D. Hilbert and W. Ackermann.²

In this more comprehensive logical theory we have two kinds of variables, individual variables and relation variables; as individual variables we use the small letters 'x', 'y', 'z', . . . , and as relation variables the capital letters 'R', 'S', 'T', We have further in our theory certain constants: first the connectives of the sentential calculus, namely the negation sign '∼', the implication sign '→', the equivalence sign '↔', the disjunction sign '∨', and the conjunction sign '∧'; secondly the two quantifiers, the universal quantifier '∏' and the existential quantifier '∑'.

From these variables and constants we may form various expressions; among these we distinguish certain expressions which we may call sentences, or rather sentential functions. Expressions of the form 'xRy' (read 'x has the relation R to y') are called elementary sentences, and we form compound sentences (as usual) by putting in front of a sentence the negation sign, or a quantifier with a subscript individual variable—e.g.,

$$\prod_x$$

² *Grundzüge der theoretischen Logik*, Berlin 1938 (second edition), p. 45 ff.

—or else by combining two simpler sentences by means of one of the signs ‘ \rightarrow ’, ‘ \leftrightarrow ’, ‘ \vee ’, ‘ \wedge ’.

In a well-known fashion we single out from among all sentences a certain class of sentences which we call axioms, we formulate further certain rules of inference such as the rules of substitution and detachment and rules concerning the use of quantifiers, and all sentences obtained from axioms by applying our rules of inference any number of times we call theorems.

We are now going to subject this theory to a certain modification, or, strictly speaking, to an extension, introducing certain constants which are specific to the calculus of relations, altogether eleven in number. These include first four relation constants, namely the symbol ‘1’ for the universal relation, the symbol ‘0’ for the null relation, the symbol ‘1’ for the identity relation (between individuals) and the symbol ‘0’ for the diversity relation. Then we have further six operation signs; namely two symbols for unary operations (on relations), the sign of the complement ‘ \sim ’ and the sign of the converse ‘ \checkmark ’; and four symbols for binary operations, the signs of addition ‘+’, multiplication ‘ \cdot ’, relative addition ‘ \dagger ’, and relative multiplication ‘ \ddagger ’. Finally we have the identity sign ‘=’, which denotes identity between relations. We shall refer to the symbols ‘1’, ‘0’, ‘ \sim ’, ‘+’, ‘ \cdot ’ and to the concepts denoted by these symbols as the absolute (or Boolean) constants and concepts; the symbols ‘1’, ‘0’, ‘ \checkmark ’, ‘ \dagger ’, ‘ \ddagger ’ and the corresponding concepts will be called the relative (or Peircean) constants and concepts.

From relation variables, relation constants, and operation signs we construct expressions of a new kind, which are called relation designations. Elementary relation designations are relation variables and relation constants. Compound relation designations are formed from simpler ones by putting symbols for unary operations above them or by joining them by means of symbols for binary operations. We obtain in this way such expressions as ‘ $\checkmark R$ ’ (read ‘the converse of R ’), ‘ $R;S$ ’ (read ‘the relative product of R and S ’), and so on.

The notion of a sentence which appeared in our original theory receives a certain extension. As elementary sentences we now take expressions of the form ‘ xRy ’ and expressions of the form ‘ $R=S$ ’, where ‘ x ’ and ‘ y ’ stand for any individual variables and ‘ R ’ and ‘ S ’ for any relation designations. The ways in which compound sentences are constructed from simpler ones remain unchanged.

To the axioms of our original theory we add as new axioms certain sentences which are intended to explain the meanings of our new constants, and which could for the most part be regarded as definitions if our original theory were provided with appropriate rules of definition. Thus we have the following new axioms:

1. $\prod_x \prod_y x1y.$
2. $\prod_x \prod_y \sim x0y.$
3. $\prod_x x1'x.$
4. $\prod_x \prod_y \prod_z [(xRy \wedge y1'z) \rightarrow xRz].$

5. $\prod_x \prod_y [x0'y \leftrightarrow (\sim x1'y)]$.
6. $\prod_x \prod_y [x\bar{R}y \leftrightarrow (\sim xRy)]$.
7. $\prod_x \prod_y [x\check{R}y \leftrightarrow yRx]$.
8. $\prod_x \prod_y [x R+S y \leftrightarrow (xRy \vee xSy)]$.
9. $\prod_x \prod_y [x R \cdot S y \leftrightarrow (xRy \wedge xSy)]$.
10. $\prod_x \prod_y [x R \uparrow S y \leftrightarrow \prod_z (xRz \vee zSy)]$.
11. $\prod_x \prod_y [x R;S y \leftrightarrow \sum_z (xRz \wedge zSy)]$.
12. $R=S \leftrightarrow \prod_x \prod_y (xRy \leftrightarrow xSy)$.

The rules of inference remain essentially unchanged, with the exception that the rule of substitution now allows the replacement of relation variables not only by other relation variables but also by any relation designations.

The theory thus outlined may be called the *elementary theory of (binary) relations*. If we confine ourselves to those sentences and theorems which contain no individual variables, we obtain the fragment of this theory in which we are here interested, namely the calculus of relations.

Let me give here some examples of theorems of this calculus:

- I. $(R = S \wedge R = T) \rightarrow S = T$.
- II. $R = S \rightarrow (R+T = S+T \wedge R \cdot T = S \cdot T)$.
- III. $R+S = S+R \wedge R \cdot S = S \cdot R$.
- IV. $(R+S) \cdot T = (R \cdot T) + (S \cdot T) \wedge (R \cdot S) + T = (R+T) \cdot (S+T)$.
- V. $R+0 = R \wedge R \cdot 1 = R$.
- VI. $R+\bar{R} = 1 \wedge R \cdot \bar{R} = 0$.
- VII. $\sim 1 = 0$.
- VIII. $\check{\bar{R}} = R$.
- IX. $\overbrace{R;S} = \check{S};\check{R}$.
- X. $R;(S;T) = (R;S);T$.
- XI. $R;1' = R$.
- XII. $R;1 = 1 \vee 1;\bar{R} = 1$.

$$\text{XIII. } (R;S) \cdot \check{T} = 0 \rightarrow (S;T) \cdot \check{R} = 0.$$

$$\text{XIV. } 0' = \bar{1}.$$

$$\text{XV. } R+S = \overline{\bar{R};\bar{S}}.$$

The proofs of these theorems present no difficulties. The simplest way is to transform the theorems (on the basis of axioms 1-12) into equivalent sentences which contain no specific constants of the calculus of relations and then to prove the resulting sentences by the usual methods of the restricted functional calculus. As the result of such a transformation of theorem XII, for instance, we obtain a simple theorem of the functional calculus,

$$\left(\prod_x \sum_y xRy \right) \vee \left(\prod_y \sum_x \sim xRy \right),$$

which is more familiar in the following form:

$$\left(\sum_y \prod_x xRy \right) \rightarrow \left(\prod_x \sum_y xRy \right).$$

(Only the symbol "1" cannot be eliminated in this way; but the only theorems in which this symbol occurs, namely theorems XI and XIV, follow almost immediately from axioms 3-5.)

The above way of constructing the elementary theory of relations will probably seem quite natural to any one who is familiar with modern mathematical logic. If, however, we are interested not in the whole theory of relations but merely in the calculus of relations, we must admit that this method has certain defects from the point of view of simplicity and elegance. We obtain the calculus of relations in a very roundabout way, and in proving theorems of this calculus we are forced to make use of concepts and statements which are outside the calculus. It is for this reason that I am going to outline another method of developing this calculus.

In constructing the calculus of relations according to the second method we use only one kind of variables, namely relation variables, and we use the same constants as in the first method, with the exception of the quantifiers. From these constants and variables we construct relation designations exactly as before. In the construction of sentences, however, certain modifications are necessary on account of the absence of individual variables and quantifiers. As elementary sentences we take only sentences of the form ' $R=S$ ', where ' R ' and ' S ' stand for relation designations; and we form compound sentences from simpler ones by means of the connectives of the sentential calculus.

Moreover we single out certain sentences which we call axioms. These can be divided into three groups.

The axioms of the first group characterize, so to speak, the meanings of the sentential connectives: although all the constants occur in them, it is only the sentential connectives which occur in them in an essential way. In order to formulate these axioms, we take any axiom system for the sentential calculus which contains as primitive terms all the sentential connectives previously

enumerated, and substitute for the variables therein arbitrary sentences of our calculus (in all possible ways).

The axioms of the second group serve to characterize the meanings of the absolute constants. We obtain these axioms by taking any set of axioms for Boolean algebra and replacing class variables by relation variables; e.g., we may for this purpose use theorems I–VII given above.³ Thus the part of the calculus of relations which involves only the absolute concepts coincides in its formal structure with Boolean algebra.

The axioms of the third group are specific to the calculus of relations, and express fundamental properties of the relative concepts. As these axioms we may take theorems VIII–XV. The first four of these involve exclusively the symbols ‘1’, ‘∩’, and ‘;’, the next two establish some connections between the absolute and relative concepts, and the last two may be considered as definitions of the symbols ‘0’ and ‘+’.

Since all the above axioms are theorems of the calculus of relations as constructed by our first method, they are undoubtedly true sentences from the point of view of the intuitive meaning which we ascribe to the symbols occurring in them.

We may, however, make it plausible in another way that these sentences are true, namely by means of a geometric representation.⁴ Let us suppose that our relation variables denote relations between real numbers, and let us consider a rectangular coordinate system in the plane. Every relation R may then be represented as a certain point set in the plane, namely as the set of all points (x, y) such that x has the relation R to y ; and conversely every point set in the plane represents a certain relation, namely the relation which holds between two numbers x and y if and only if the point (x, y) belongs to the set. The formula ‘ $R = S$ ’ is valid in this geometrical representation if and only if the sets corresponding to R and S are identical. The relations 1, 0, 1’, 0’, are represented by certain particular point sets: viz., 1 by the whole plane, 0 by the empty point set, the identity relation by the straight line whose equation is ‘ $x = y$ ’, the diversity relation by the set of all points not on this straight line. To the operations on relations there correspond certain operations on point sets. The absolute operations are represented by the usual set-theoretic operations on point sets: $R + S$ by the union of sets, $R \cdot S$ by the intersection, \bar{R} by the complement. In order to obtain the representation for \bar{R} , we take the point set corresponding to R and rotate it (in three dimensions) through an angle of 180° about the line $x = y$. Unfortunately it is more difficult to explain the meaning of the geometrical operations which correspond to relative addition and relative multi-

³ This part of our axiom system is due essentially to E. V. Huntington; see his paper *Sets of independent postulates for the algebra of logic*, *Transactions of the American Mathematical Society*, vol. 5 (1904), p. 292 f. It is easily seen that axioms I–VII are symmetric with respect to the absolute constants ‘+’ and ‘·’, as well as ‘1’ and ‘0’. It would be desirable to have a simple and independent axiom system for the whole calculus of relations which would be symmetric with respect to both absolute and relative constants.

⁴ Cf. Schröder, *op. cit.*, p. 52 ff.; C. Kuratowski and A. Tarski, *Les opérations logiques et les ensembles projectifs*, *Fundamenta mathematicae*, vol. 17 (1931), p. 240 ff.

plication. We have to resort to three-dimensional space and to supplement our coordinate system with a z -axis perpendicular to the xy -plane. Then, in order to obtain the representation of $R;S$ (in the xy -plane), we rotate the point set corresponding to R through a right angle about the x -axis, draw through every point of the resulting set a straight line parallel to the y -axis, and take the union of all these straight lines; in this way we obtain the "cylindrical" point set R^* . Similarly we rotate the set corresponding to S through a right angle about the y -axis, draw the lines parallel to the x -axis, and obtain the "cylindrical" point set S^* . Finally we take the intersection of R^* and S^* , and project it orthogonally upon the xy -plane. The projection thus obtained constitutes the geometrical representation of $R;S$. The representation of the relative sum can easily be obtained from that of the relative product, in view of axiom XV. The construction becomes much simpler in the case that one of the terms of the relative sum or the relative product is the universal relation or the null relation. For example, it is easily seen that the relation $R;1$ holds between x and y if and only if there is a z such that xRz (so that it is, so to speak, independent of y); therefore, in order to obtain the geometrical representation of $R;1$, we consider the set corresponding to R , draw through every point of this set a straight line parallel to the y -axis, and take the union of all these straight lines. The same method is applied in the case of $1;R$, except that the straight lines are drawn parallel to the x -axis.

With the aid of this geometric representation, the content of all our axioms for the calculus of relations is easily made intuitively evident. Thus, e.g., axiom VIII corresponds to the intuitively obvious geometric fact that, if we take any point set and rotate it through an angle of 180° about a given straight line, and then rotate it again through 180° about the same straight line, the result is the original point set.

Let us now consider axiom XII. We have to show that either the set representing $R;1$ or the set representing $1;\bar{R}$ is the whole plane. Suppose that the set representing $R;1$ is not the whole plane. As we have seen, this set is the union of all straight lines which are parallel to the y -axis and which pass through a point belonging to the set which represents R . If this union does not coincide with the whole plane, there must be a line parallel to the y -axis which does not contain any point of the set corresponding to R , and which therefore consists exclusively of points of the set corresponding to \bar{R} . If through every point of this line we draw a straight line parallel to the x -axis and take the union of all these parallel lines, we obtain the whole plane. Hence the relation $1;\bar{R}$ is represented by the whole plane.

Continuing our account of the development of the calculus of relations according to the second method, we have to describe the way in which theorems of the calculus of relations are to be derived from our axioms. Here we proceed as in the case of the first method; i.e., we formulate rules of inference, and we call theorems those sentences which can be derived from axioms by applying the rules of inference. But the situation is now simpler in that we are able to restrict ourselves to two rules of inference—the rule of substitution and the rule of detachment.

In order to illustrate the technique of this domain, we shall outline here the proofs of several theorems of the calculus of relations. In these proofs we presuppose familiarity with the sentential calculus and with Boolean algebra, and we shall apply laws of these two theories without explicitly indicating them.

$$\text{XVI. } [(R;S) \cdot T = 0 \leftrightarrow (S;\check{T}) \cdot \check{R} = 0] \wedge [(R;S) \cdot T = 0 \leftrightarrow (\check{T};R) \cdot \check{S} = 0].$$

Proof. In XIII substitute ' \check{T} ' for ' T '. This gives:

$$(1) \quad (R;S) \cdot \check{T} = 0 \rightarrow (S;\check{T}) \cdot \check{R} = 0.$$

Again in XIII substitute ' S ', ' \check{T} ', and ' R ' for ' R ', ' S ', and ' T ' respectively, so obtaining:

$$(2) \quad (S;\check{T}) \cdot \check{R} = 0 \rightarrow (\check{T};R) \cdot \check{S} = 0.$$

Applying XIII for the third time, with ' R ', ' S ', and ' T ' replaced by ' \check{T} ', ' R ', and ' S ' respectively, we obtain:

$$(3) \quad (\check{T};R) \cdot \check{S} = 0 \rightarrow (R;S) \cdot \check{T} = 0.$$

Now (1)–(3) imply:

$$(4) \quad [(R;S) \cdot \check{T} = 0 \leftrightarrow (S;\check{T}) \cdot \check{R} = 0] \wedge [(R;S) \cdot \check{T} = 0 \leftrightarrow (\check{T};R) \cdot \check{S} = 0].$$

On the other hand we have by VIII:

$$(5) \quad \check{T} = T.$$

From (4) and (5), theorem XVI follows directly.

$$\text{XVII. } R \cdot \bar{S} = 0 \rightarrow \check{R} \cdot \bar{\check{S}} = 0.$$

Proof. By XI we have:

$$(6) \quad \check{R};1' = \check{R} \wedge \check{S};1' = \check{S}.$$

Hence obviously:

$$(7) \quad (\check{S};1') \cdot \bar{\check{S}} = 0.$$

If now in XVI we replace ' R ', ' S ', and ' T ' by ' \check{S} ', ' $1'$ ', and ' $\bar{\check{S}}$ ' respectively, we obtain, by (7):

$$(8) \quad (1';\bar{\check{S}}) \cdot \check{S} = 0.$$

By VIII we have:

$$(9) \quad R \cdot \bar{S} = 0 \rightarrow \check{R} \cdot \bar{\check{S}} = 0.$$

By virtue of a law of Boolean algebra, (8) and (9) imply:

$$(10) \quad R \cdot \bar{S} = 0 \rightarrow (1';\bar{\check{S}}) \cdot \check{R} = 0.$$

Applying XVI again, with ' R ', ' S ', and ' T ' replaced by ' \check{R} ', ' $1'$ ', and ' $\bar{\check{S}}$ ' respectively, we obtain:

$$(11) \quad (\check{R};1') \cdot \bar{\check{S}} = 0 \leftrightarrow (1';\bar{\check{S}}) \cdot \check{R} = 0.$$

From (10) and (11) follows:

$$(12) \quad R \cdot \bar{S} = 0 \rightarrow (\check{R}; 1') \cdot \bar{\check{S}} = 0.$$

From this together with (6) we obtain XVII immediately.

$$\text{XVIII. } R = S \rightarrow \check{R} = \check{S}$$

Proof. We have the following formulae (where (13) is a well-known law of Boolean algebra, (14) follows from XVII, and (15) is a particular case of (13)):

$$(13) \quad R = S \leftrightarrow (R \cdot \bar{S} = 0 \wedge S \cdot \bar{R} = 0).$$

$$(14) \quad (R \cdot \bar{S} = 0 \wedge S \cdot \bar{R} = 0) \rightarrow (\check{R} \cdot \bar{\check{S}} = 0 \wedge \check{S} \cdot \bar{\check{R}} = 0).$$

$$(15) \quad \check{R} = \check{S} \leftrightarrow (\check{R} \cdot \bar{\check{S}} = 0 \wedge \check{S} \cdot \bar{\check{R}} = 0).$$

From these XVIII follows directly.

$$\text{XIX. } \overbrace{R+S} = \check{R} + \check{S}.$$

Proof. For simplicity we replace ' $\check{R} + \check{S}$ ' by ' T '. Since obviously

$$(16) \quad \check{R} \cdot \bar{T} = 0 \wedge \check{S} \cdot \bar{T} = 0,$$

we infer by XVII that

$$(17) \quad \check{\check{R}} \cdot \bar{\check{T}} = 0 \wedge \check{\check{S}} \cdot \bar{\check{T}} = 0.$$

Hence by virtue of VIII,

$$(18) \quad R \cdot \bar{\check{T}} = 0 \wedge S \cdot \bar{\check{T}} = 0,$$

and consequently,

$$(19) \quad (R + S) \cdot \bar{\check{T}} = 0.$$

If now in XVII we replace ' R ' by ' $R + S$ ' and ' T ' by ' \check{T} ', we obtain, by (19),

$$(20) \quad \overbrace{R+S} \cdot \bar{\check{T}} = 0,$$

and hence, by VIII,

$$(21) \quad \overbrace{R+S} \cdot \bar{T} = 0, \text{ i.e., } \overbrace{R+S} \cdot \overline{\check{R} + \check{S}} = 0.$$

Since, on the other hand,

$$(22) \quad R \cdot \overline{R+S} = 0 \wedge S \cdot \overline{R+S} = 0,$$

we obtain, again using XVII,

$$(23) \quad \check{R} \cdot \overline{R+S} = 0 \wedge \check{S} \cdot \overline{R+S} = 0,$$

and hence,

$$(24) \quad (\check{R} + \check{S}) \cdot \overline{R+S} = 0.$$

From (21) and (24), theorem XIX follows (cf. formula (13) in the proof of XVIII).

$$\text{XX. } \check{0} = 0 \wedge \check{1} = 1.$$

Proof. Applying XVII twice, the first time with 'R' replaced by '0' and 'S' by ' $\check{0}$ ', and the second time with 'R' replaced by '1' and 'S' by '1', we obtain:

$$(25) \quad \check{0}.\check{0} = 0 \wedge \check{1}.\check{1} = 0.$$

Hence by VIII:

$$(26) \quad \check{0}.\bar{0} = 0 \wedge 1.\bar{1} = 0.$$

Hence further:

$$(27) \quad \check{0}.1 = 0 \wedge \bar{1} = 0.$$

And from this, XX follows at once.

$$\text{XXI. } S.\bar{T} = 0 \rightarrow (R;S) \cdot \overline{R;T} = 0.$$

Proof. We have obviously:

$$(28) \quad (R;T) \cdot \overline{R;T} = 0.$$

In XVI replace 'S' by 'T', and 'T' by ' $\overline{R;T}$ '. By (28) we obtain:

$$(29) \quad (\overline{\overline{R;T}}; R) \cdot \check{T} = 0.$$

By XVII we have:

$$(30) \quad S.\bar{T} = 0 \rightarrow \check{S}.\check{T} = 0.$$

Formulae (29), (30) imply:

$$(31) \quad S.\bar{T} = 0 \rightarrow (\overline{\overline{R;T}}; R) \cdot \check{S} = 0.$$

Applying XVI again, with ' $R;T$ ' put in place of 'T', we obtain:

$$(32) \quad (R;S) \cdot \overline{R;T} = 0 \leftrightarrow (\overline{\overline{R;T}}; R) \cdot \check{S} = 0.$$

Then XXI follows immediately from (31) and (32).

$$\text{XXII. } S = T \rightarrow R;S = R;T.$$

Proof. XXII can be derived from XXI in exactly the same way as XVIII from XVII.

$$\text{XXIII. } R;(S+T) = (R;S)+(R;T).$$

Proof. For brevity we replace ' $(R;S)+(R;T)$ ' by 'U'. We have obviously:

$$(33) \quad (R;S) \cdot \bar{U} = 0 \wedge (R;T) \cdot \bar{U} = 0.$$

Hence by XVI we obtain:

$$(34) \quad (\check{U};R) \cdot \check{S} = 0 \wedge (\check{U};R) \cdot \check{T} = 0.$$

Consequently:

$$(35) \quad (\check{U};R) \cdot (\check{S} + \check{T}) = 0.$$

Since, by XIX,

$$(36) \quad \overbrace{S+T} = \check{S} + \check{T},$$

we infer from (35) that:

$$(37) \quad (\check{U};R) \cdot \overbrace{S+T} = 0.$$

Now in XVI replace 'S' by 'S+T', and 'T' by 'U'. Using (37), we obtain:

$$(38) \quad [R;(S+T)] \cdot \bar{U} = 0, \text{ i.e., } [R;(S+T)] \cdot \overline{(R;S)+(R;T)} = 0.$$

On the other hand we have:

$$(39) \quad S \cdot \overline{S+T} = 0 \wedge T \cdot \overline{S+T} = 0.$$

Hence by XXI:

$$(40) \quad (R;S) \cdot \overline{R;(S+T)} = 0 \wedge (R;T) \cdot \overline{R;(S+T)} = 0.$$

Hence further:

$$(41) \quad [(R;S)+(R;T)] \cdot \overline{R;(S+T)} = 0.$$

From (38) and (41), XXIII follows directly (cf. formula (13) in the proof of XVIII).

$$\text{XXIV. } R;0 = 0.$$

Proof. By XX we have:

$$(42) \quad (\check{1};R) \cdot \check{0} = 0.$$

Hence, by means of XVI (with 'S' replaced by '0' and 'T' by '1'):

$$(43) \quad (R;0) \cdot 1 = 0.$$

Then XXIV is an immediate consequence of (43).

Proofs of the four following theorems are entirely analogous to those of XXI-XXIV:

$$\text{XXV. } R \cdot \check{S} = 0 \rightarrow (R;T) \cdot \overline{S;T} = 0.$$

$$\text{XXVI. } R = S \rightarrow R;T = S;T.$$

$$\text{XXVII. } (R+S);T = (R;T)+(S;T).$$

$$\text{XXVIII. } 0;R = 0.$$

$$\text{XXIX. } \check{1}' = 1'.$$

Proof. By IX and XI we have:

$$(44) \quad \overline{\check{R};1'} = \check{1}';\check{R}$$

and

$$(45) \quad \check{R};1' = \check{R}.$$

Applying XVIII to (45), we get:

$$(46) \quad \overline{\check{R};1'} = \check{R}.$$

Hence by (44):

$$(47) \quad \check{1}';\check{R} = \check{R}.$$

With the aid of VIII and XXII we obtain from (47):

$$(48) \quad \check{1}';R = R.$$

Substituting here '1' for 'R', we get:

$$(49) \quad \check{1}';1' = 1'.$$

On the other hand we have, by XI:

$$(50) \quad \check{1}';1' = \check{1}'.$$

And XXIX follows directly from (49) and (50).

XXX. $1';R = R.$

Proof. Applying XXVI to XXX, we obtain:

$$(51) \quad \check{1}';R = 1';R.$$

This formula together with the formula (48) (see the proof of XXIX) yields XXX directly.

XXXI. $(1;S) \cdot T = 0 \rightarrow (1;T) \cdot S = 0.$

Proof. By XVI we have:

$$(52) \quad (1;S) \cdot T = 0 \rightarrow (\check{T};1) \cdot \check{S} = 0.$$

By applying XXII to XX we obtain:

$$(53) \quad \check{T};1 = \check{T};\check{1}.$$

On the other hand, IX gives:

$$(54) \quad \overline{1;T} = \check{T};\check{1}.$$

Formulae (52)–(54) imply:

$$(55) \quad (1;S) \cdot T = 0 \rightarrow \overline{1;T} \cdot \check{S} = 0.$$

In what follows, '(1;T)·S' will be replaced by 'U'. We have obviously:

$$(56) \quad U \cdot \overline{1;T} = 0 \wedge U \cdot \overline{S} = 0.$$

Hence, applying XVII twice, we obtain:

$$(57) \quad \check{U} \cdot \overline{1;T} = 0 \wedge \check{U} \cdot \overline{S} = 0.$$

In consequence of laws of Boolean algebra, (57) leads to:

$$(58) \quad \check{U} \cdot \overline{1;T \cdot S} = 0.$$

Hence:

$$(59) \quad \overline{1;T} \cdot \check{S} = 0 \rightarrow \check{U} = 0.$$

By XVIII we have:

$$(60) \quad \check{U} = 0 \rightarrow \check{\check{U}} = \check{0}.$$

Hence in view of VIII and XX:

$$(61) \quad \check{U} = 0 \rightarrow U = 0, \text{ i.e., } \check{U} = 0 \rightarrow (1;T) \cdot S = 0.$$

Formulae (55), (59), (61) together yield XXXI.

$$\text{XXXII. } (\sim R = 1) \leftrightarrow (1;\bar{R});1 = 1.$$

Proof. Since

$$(62) \quad R = 1 \rightarrow \bar{R} = 0,$$

we have, by XXII,

$$(63) \quad R = 1 \rightarrow 1;\bar{R} = 1;0,$$

and hence by XXIV,

$$(64) \quad R = 1 \rightarrow 1;\bar{R} = 0.$$

From (64) we obtain, using XXVI,

$$(65) \quad R = 1 \rightarrow (1;\bar{R});1 = 0;1,$$

and hence by XXVIII,

$$(66) \quad R = 1 \rightarrow (1;\bar{R});1 = 0.$$

From (66) results obviously:

$$(67) \quad (1;\bar{R});1 = 1 \rightarrow (\sim R = 1).$$

On the other hand we obtain from XII, replacing 'R' by '1;R':

$$(68) \quad (1;\bar{R});1 = 1 \vee 1; \overline{1;\bar{R}} = 1.$$

From XXXI, substituting 'R' for 'S' and '1;R' for 'T', we infer:

$$(69) \quad (1; \overline{1;\bar{R}}) \cdot \bar{R} = 0.$$

Hence:

$$(70) \quad 1; \overline{1; \bar{R}} = 1 \rightarrow \bar{R} = 0.$$

And hence:

$$(71) \quad 1; \overline{1; \bar{R}} = 1 \rightarrow R = 1.$$

Then (68) and (71) together imply:

$$(72) \quad (\sim R = 1) \rightarrow (1; \bar{R}); 1 = 1.$$

And XXXII follows immediately from (67) and (72).

The theorem just proved plays a very important rôle in the calculus of relations, since it allows us to prove the following general metalogical theorem:

Every sentence of the calculus of relations can be transformed into an equivalent sentence of the form 'R = S', and even of the form 'T = 1'.⁵

In fact, we may first, on the basis of a well-known theorem of Boolean algebra, transform every equation

$$R = S$$

into an equation of the form

$$T = 1,$$

namely into

$$R \cdot S + \bar{R} \cdot \bar{S} = 1.$$

Now, on the basis of theorem XXXII, we may transform the negation of an equation of the form 'T = 1', i.e.,

$$\sim T = 1,$$

into an equation of the form 'T = 1', namely

$$1; \bar{T}; 1 = 1.$$

Further we are able to transform a conjunction of two such equations, i.e.,

$$T = 1 \wedge U = 1,$$

into one equation, namely

$$T \cdot U = 1.$$

Since it is known from the sentential calculus that it is possible to eliminate from every sentence all sentential connectives except the signs of negation and conjunction, it follows from the above transformations that our metalogical theorem is valid.

This metalogical theorem suggests still another way of constructing the calculus of relations. For it shows that we may confine ourselves, in developing

⁵ See Schröder, op. cit., p. 153.

this calculus, to sentences which have the form of equations (or which even have the form ' $T=1$ '), thus dispensing with the concepts and theorems of the sentential calculus. For this purpose we should have to put all our axioms into the form of equations, and to give rules which would permit us to derive new equations from given ones. Though this plan has not been worked out in detail, the realization of it presents no essential difficulty.

In the further development of the calculus of relations we may introduce new concepts definable with the help of the fundamental concepts of the calculus. We may, for instance, distinguish certain especially important categories of relations, such as symmetric relations, transitive relations, ordering relations, one-many relations or functions, and one-one relations or biunique functions. In this connection the following point deserves special attention. It may be noticed that many laws of the calculus of relations, and in particular the axioms VIII–XI adopted under our second method, resemble theorems of the theory of groups, relative multiplication playing the rôle of group-theoretic composition, and the converse of a relation corresponding to the group-theoretic inverse. Let us now consider briefly the relations satisfying the condition:

$$R;\check{R} = 1' \wedge \check{R};R = 1'.$$

This condition expresses in ordinary mathematical terminology that the relation R maps the class of individuals on itself in a one-to-one manner. If we confine ourselves to relations satisfying this condition, we can easily prove that they satisfy all the axioms of the theory of groups. Thus it turns out that the calculus of relations includes the elementary theory of groups and is, so to speak, a union of Boolean algebra and group theory. This fact accounts for the deductive power and mathematical richness of the calculus.

I should like now to discuss some metalogical problems regarding the calculus of relations.

The first problem concerns the relation between our two methods of constructing the calculus. It is obvious that every theorem which can be proved by the second method can also be proved by the first method (since, as mentioned before, the axioms adopted under the second method can be proved as theorems under the first method, and all the rules of inference used in the second method were also assumed in the first method). It is, however, by no means obvious that the converse is also true, and that consequently our two methods are entirely equivalent. Since it follows from a result of K. Gödel⁶ that every sentence which is true in every domain of individuals is provable by our first method, the problem can also be put in the following way: Is it the case that every sentence of the calculus of relations which is true in every domain of individuals is derivable from the axioms adopted under the second method? This problem presents some difficulties and still remains open. I can only say that I am practically sure that I can prove with the help of the second method

⁶ *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, *Monatshefte für Mathematik und Physik*, vol. 37 (1930), pp. 349–360.

all of the hundreds of theorems to be found in Schröder's *Algebra und Logik der Relative*.

The next problem is the so-called representation problem. Is every model of the axiom system of the calculus of relations isomorphic with a class of binary relations which contains the relations 1, 0, 1', 0' and is closed under all the operations considered in this calculus? As is known, the analogous problem for Boolean algebra was raised, and solved in the affirmative, by M. H. Stone.⁷ For the calculus of relations the problem remains open; a particular case of it—for an atomistic system of the theory of relations—was recently solved by J. C. C. McKinsey.⁸ The two problems—that of the equivalence of our two construction methods, and the representation problem (in its application to the axiom system adopted under the second method)—are related to each other: from an affirmative solution of the second problem we could obtain an affirmative solution of the first.

In discussing the remaining metalogical problems we shall have in mind our first method of constructing the calculus of relations.

The first of these problems is known as the decision problem. It can be formulated in the following way: Is there a method which would enable us in every particular case to decide whether a given sentence expressed in the terms of the calculus of relations is a theorem of this calculus? We know from a result of A. Church⁹ that no such method exists with regard to what we called the elementary theory of relations—i.e., with regard to sentences which contain not only relation variables but also individual variables. With the aid of this result it can be shown that the solution of the decision problem is likewise negative for the calculus of relations proper.¹⁰

The next problem concerns the connection between the elementary theory of relations—or, what is practically the same, the restricted functional calculus—and the calculus of relations. Since the calculus of relations is only a proper part of the theory of relations, the problem arises whether every sentence formulated in the elementary theory of relations and concerned essentially only with the properties of relations (i.e., containing only relation variables as free variables) can be transformed into an equivalent sentence of the calculus of relations. In a less exact form this problem can be put as follows: Is it true that every property of relations, relation between relations, operation on relations, etc. which can be defined in the restricted functional calculus can be expressed also in the calculus of relations? It was shown by A. Korselt that the answer to this question is negative.¹¹ His proof, however, depends essentially on admitting

⁷ *The theory of representations for Boolean algebras*, *Transactions of the American Mathematical Society*, vol. 40 (1936), pp. 37–111; see in particular p. 106.

⁸ *Postulates for the calculus of binary relations*, this JOURNAL, vol. 5 (1940), pp. 85–97; see in particular p. 94.

⁹ *A note on the Entscheidungsproblem*, this JOURNAL, vol. 1 (1936), pp. 40–41 (and *Correction*, *ibid.*, pp. 101–102).

¹⁰ The proof will be given later in a separate paper.

¹¹ Korselt's result was published in the paper of L. Löwenheim, *Über Möglichkeiten im Relativkalkül*, *Mathematische Annalen*, vol. 76 (1915), pp. 447–470.

finite domains of individuals with various numbers of elements. I have shown that the result remains valid if we confine ourselves to infinite domains of individuals. It follows from my reasoning that even the properties of relations expressed by such simple formulae of the restricted functional calculus as

$$\prod_x \prod_y \prod_z \sum_u (xRu \wedge yRu \wedge zRu)$$

or

$$\sum_x \sum_y \sum_z \sum_u (xRy \wedge xRz \wedge xRu \wedge yRz \wedge yRu \wedge zRu)$$

cannot be expressed in the terms of the calculus of relations; i.e., no sentence of the calculus of relations is satisfied by exactly the same binary relations (between elements of an infinite set) which satisfy either one of the above formulae. Moreover, I have succeeded in considerably extending this result. For I have shown that the answer remains negative even if we enrich the calculus of relations by the addition of any finite number of new constants denoting fixed relations, properties of relations, operations on relations, and so on, provided that all concepts introduced in this way are invariant under any one-to-one mapping of the class of individuals on itself.¹⁰ (It follows from an earlier result by A. Lindenbaum and myself¹² that this invariance property belongs to all concepts definable within the restricted functional calculus, or even within much more comprehensive logical systems, such as that of *Principia mathematica*.)

Our last metalogical problem is closely related to the preceding one. Since there are sentences of the elementary theory of relations which cannot be transformed into equivalent sentences of the calculus of relations, we may look for a criterion which would enable us to decide in every particular case whether such a transformation is possible. We are here confronted with a new decision problem. This problem is so far unsolved, but it seems plausible that its solution is negative.

The aim of this paper has been, not so much to present new results, as to awaken interest in a certain neglected logical theory, and to formulate some new problems concerning this theory. I do believe that the calculus of relations deserves much more attention than it receives. For, aside from the fact that the concepts occurring in this calculus possess an objective importance and are in these times almost indispensable in any scientific discussion, the calculus of relations has an intrinsic charm and beauty which makes it a source of intellectual delight to all who become acquainted with it.¹³

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¹² *Über die Beschränktheit der Ausdrucksmittel deduktiver Theorien, Ergebnisse eines mathematischen Kolloquiums*, no. 7 (1936), pp. 15–22. (There are several misprints in that paper. In particular, read ‘metamathematisch’ and ‘Metamathematik’ instead of ‘mathematisch’ and ‘Mathematik.’)

¹³ I wish to take this opportunity to express my gratitude to Dr. J. C. C. McKinsey for his assistance in preparing this paper.